

# Some extensions of Hardy's integral inequalities to Hardy type spaces

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## Abstract

In this paper some extensions of Hardy's integral inequalities to  $0 < p \leq 1$  are established.

## 1. Introduction

Let

$$Hf(x) = \frac{1}{x} \int_0^x f(t)dt, (x > 0),$$

and (dual form)

$$H^*f(x) = \int_x^\infty f(t)dt, (x > 0).$$

The Hardy integral inequality can be stated as (see [3]):

$$\int_0^\infty (Hf(x))^p dx < p'^p \int_0^\infty (f(x))^p dx, p > 1, f(x) \geq 0, \quad (1)$$

$$\int_0^\infty (H^*f(x))^p dx < p^p \int_0^\infty (xf(x))^p dx, p > 1, f(x) \geq 0, \quad (2)$$

unless  $f \equiv 0$ .

The inequality (1) was firstly proved by Hardy [4], but the constant was not determined; and Landau found out precisely the constant is  $p'^p$  in [7]; later, Hardy [5] generalized it to the inequality (2) himself. However, if  $0 < p < 1$ , the reverse direction inequalities hold (see [3]):

$$\int_0^\infty (Hf(x))^p dx > p'^p \int_0^\infty (f(x))^p dx, 0 < p < 1, f(x) \geq 0, \quad (3)$$

$$\int_0^\infty (H^*f(x))^p dx > p^p \int_0^\infty (xf(x))^p dx, 0 < p < 1, f(x) \geq 0. \quad (4)$$

unless  $f \equiv 0$ . The constants in (1),(2), (3) and (4) are the best possible.

The positive direction inequalities (1) and (2) play an important role in many areas such as harmonic analysis [12], PDE [8], etc. In view of this, much efforts and time have been devoted to their improvement and generalizations over the years.

When  $p > 1$ , many generalizations include the works in numerous papers, for example, [3,11,6] and some of the references cited therein.

When  $p = 1$ , in view of the theory of Hardy spaces on  $\mathbf{R}^n$  established by Coifman and Weiss in [1] and some others, J. Garcia-Cuerva and J. L. Rubio de Francia extended (1) to Hardy spaces, and established a positive direction inequality of Hardy type for  $p = 1$ : let  $f \in H^1(\mathbf{R})$  be supported in  $[0, \infty)$ , then

$$\int_0^\infty |Hf(x)|dx < (\log 2) \|f\|_{H_{at}^{1,\infty}(\mathbf{R})}, \quad (5)$$

where  $H_{at}^{1,\infty}$  is the atom Hardy spaces. See [2].

In this paper we extend the positive direction inequalities (1), (2) and (5) to  $0 < p \leq 1$ , as well as establish some estimates of  $H$  and  $H^*$  from Hardy type spaces to Hardy type spaces.

Let us introduce some definitions of the Hardy type spaces.

**Definition 1** Let  $0 < p \leq 1 \leq q \leq \infty$ ,  $p < q$  and  $s \in \mathbf{N}$  and  $w \geq 0$  is a weight function on  $\mathbf{R}^+$ .

(a) A function  $a(x)$  on  $\mathbf{R}^+$  is said to be a  $(p, q, s)_w$ -atom, if

(i)  $\text{supp } a \subset (x_0, x_1) \subset \mathbf{R}^+$ ,  $x_0 > 0$ ,

(ii)  $\|a\|_{L_w^q(\mathbf{R}^+)} \leq \left( \int_{x_0}^{x_1} w(x)dx \right)^{1/q-1/p}$ ,

(iii)  $\int_{\mathbf{R}^+} a(x)x^\beta dx = 0$ ,  $\beta = 0, 1, \dots, s$ ,

(b) and  $a(x)$  is said to be a  $L - (p, q, s)_w$ -atom, if it is a  $(p, q, s)_w$ -atom and satisfies

(iv)  $\int_{\mathbf{R}^+} a(x)\ln x dx = 0$ .

**Definition 2** Let  $p, q, s$  and  $w$  as in Definition 1. Some Hardy spaces on  $\mathbf{R}^+$  are defined by

$H_w^{p,q,s}(\mathbf{R}^+) = \{f : f = \sum_{k=1}^\infty \lambda_k a_k, \text{ where each } a_k \text{ is a } (p, q, s)_w\text{-atom, } \sum_{k=1}^\infty |\lambda_k|^p < +\infty, \text{ and the series converges in the sense of distributions}\},$

and

$LH_w^{p,q,s}(\mathbf{R}^+) = \{f : f = \sum_{k=1}^\infty \lambda_k a_k, \text{ where each } a_k \text{ is a } L - (p, q, s)_w\text{-atom, } \sum_{k=1}^\infty |\lambda_k|^p < +\infty, \text{ and the series converges in the sense of distributions}\}.$

And define the quasinorms of a function of  $H_w^{p,q,s}(\mathbf{R}^+)$  or  $LH_w^{p,q,s}(\mathbf{R}^+)$  by

$$\|f\|_{H_w^{p,q,s}(\mathbf{R}^+)} = \inf \left( \sum_{k=1}^\infty |\lambda_k|^p \right)^{1/p}, \quad \text{or} \quad \|f\|_{LH_w^{p,q,s}(\mathbf{R}^+)} = \inf \left( \sum_{k=1}^\infty |\lambda_k|^p \right)^{1/p}$$

respectively, where the infimum is taken over all the decompositions of  $f$  as above.

Simply, we denote  $H_1^{p,q,s}(\mathbf{R}^+)$  and  $LH_1^{p,q,s}(\mathbf{R}^+)$  by  $H^{p,q,s}(\mathbf{R}^+)$  and  $LH^{p,q,s}(\mathbf{R}^+)$  respectively.

$H^{p,q,0}(\mathbf{R}^+)$  is an example of the Hardy spaces on spaces of homogeneous type studied by Coifman and Weiss [1], and A. Marcias and C. Segovia [9,10].

Throughout this paper, we denote  $p' = p/(p-1)$  for  $1 < p < \infty$ ,  $\infty' = 1$ , and  $1' = \infty$ , and suppose all functions are not identical to zero.

We extend (1) and (5) to the case of  $0 < p \leq 1$ :

**Theorem 1** Let  $0 < p \leq 1 \leq q \leq \infty$  and  $p < q$ . If  $f$  is in  $H^{p,q,0}(\mathbf{R}^+)$ , then,

$$\|Hf\|_{L^p(\mathbf{R}^+)} < \frac{1}{(1-p/q)^{1/p}} \|f\|_{H^{p,q,0}(\mathbf{R}^+)}. \quad (6)$$

We extend (2) as well:

**Theorem 2** Let  $0 < p \leq 1$ . If  $f$  is in  $H_{x^p}^{p,q,0}(\mathbf{R}^+)$ , then,

$$\|Hf\|_{L^p(\mathbf{R}^+)} < \begin{cases} \|f\|_{H_{x^p}^{p,q,0}(\mathbf{R}^+)}, & \text{if } q = \infty, \\ \frac{1}{(1-p)^{1/p}(p+1)} \|f\|_{H_{x^p}^{p,q,0}(\mathbf{R}^+)}, & \text{if } q = 1, p \neq 1, \\ \frac{1}{(1-pq'/q)^{1/q'}} \frac{(1+p)^{1/q-1/p}}{((1/q'-p/q)p+1)^{1/p}} \|f\|_{H_{x^p}^{p,q,0}(\mathbf{R}^+)}, & \text{if } 1 < q < \infty, p < q-1. \end{cases} \quad (7)$$

We also extend these results to the estimates of  $H$  and  $H^*$  from Hardy type spaces to Hardy type spaces.

**Theorem 3** Let  $0 < p \leq 1 < q \leq \infty$  and  $s \in \mathbf{N}$ . If  $f$  is in  $H^{p,q,s}(\mathbf{R}^+)$ , then,

$$\|Hf\|_{H^{p,q,s}(\mathbf{R}^+)} \leq q' \|f\|_{LH^{p,q,s}(\mathbf{R}^+)}. \quad (8)$$

**Theorem 4** Let  $0 < p \leq 1 \leq q \leq \infty, s \in \mathbf{N}$  and  $s-1 > 0$ . If  $f$  is in  $H^{p,q,s}(\mathbf{R}^+)$ , then,

$$\|H^*f\|_{H^{p,q,s-1}(\mathbf{R}^+)} \leq \begin{cases} (1+p)^{1/p} \|f\|_{H^{p,q,s}(\mathbf{R}^+)}, & \text{if } q = \infty, \\ \frac{1}{(1-p)(1+p)^{1-1/p}} \|f\|_{H^{p,q,s}(\mathbf{R}^+)}, & \text{if } q = 1, p \neq 1, \\ q \|f\|_{H^{p,q,s}(\mathbf{R}^+)}, & \text{if } 1 < q < \infty, 0 < p \leq 1. \end{cases} \quad (9)$$

## 2. Proof of Theorems

Firstly, let us introduce two inequalities. By Minkowski inequalities [3], it is easy to see that

$$(x_1 - x_0)^{p+1} < x_1^{p+1} - x_0^{p+1}; \quad (10)$$

when  $x_1 > x_0 > 0$  and  $p > 0$ , and

$$(x_1 - x_0)^{1-p} > x_1^{1-p} - x_0^{1-p}. \quad (11)$$

when  $x_1 > x_0 > 0$  and  $1 > p > 0$ .

**Proof of Theorem 1** It suffices to prove the following propositions.

**Proposition 1** Let  $0 < p \leq 1 \leq q \leq \infty$  and  $p < q$ . We have

$$\int_0^\infty |Ha(x)|^p dx < \frac{1}{1-p/q} \quad (12)$$

for all  $(p, q, 0)$ -atom  $a$  on  $\mathbf{R}^+$ .

Let  $FH^{p,q,0}(\mathbf{R}^+)$  be the set of all finite linear combination of  $(p, q, 0)$ -atoms.

From Proposition 1 it is easy to get that

**Proposition 2** Let  $0 < p \leq 1 \leq q \leq \infty$  and  $p < q$ . Then (6) holds for  $f$  in  $FH^{p,q,0}(\mathbf{R}^+)$ .

**Proposition 3** Let  $0 < p \leq 1 \leq q \leq \infty$  and  $p < q$ . Then

$$Hf(x) = \sum_{k=1}^{\infty} \lambda_k Ha_k(x) \quad \text{a.e.}$$

for all  $f = \sum_{k=1}^{\infty} \lambda_k a_k \in H^{p,q,0}(\mathbf{R}^+)$ , where each  $a_k$  is a  $(p, q, 0)$ -atom and  $\sum_{k=1}^{\infty} |\lambda_k|^p < \infty$ .

In fact, once Proposition 1 and Proposition 3 have been proved, then, for  $f \in H^{p,q,0}(\mathbf{R}^+)$ , i.e.  $f = \sum_{k=1}^{\infty} \lambda_k a_k$ , we have

$$\begin{aligned}
\int_0^{\infty} |Hf(x)|^p dx &= \int_0^{\infty} \left| \sum_{k=1}^{\infty} \lambda_k H a_k(x) \right|^p dx \\
&\leq \int_0^{\infty} \sum_{k=1}^{\infty} |\lambda_k|^p |H a_k(x)|^p dx \\
&\leq \sum_{k=1}^{\infty} |\lambda_k|^p \int_0^{\infty} |H a_k(x)|^p dx \\
&\quad \text{(by Minkowski inequality)} \\
&< \frac{1}{1-p/q} \sum_{k=1}^{\infty} |\lambda_k|^p,
\end{aligned}$$

From this, (6) follows easily. Thus, we finish the proof of Theorem 1.

**Proof of Proposition 1** Let  $a$  be a  $(p, q, 0)$ -atom on  $\mathbf{R}^+$ , i.e.  $\text{supp } a \subset (x_0, x_1) \subset \mathbf{R}^+$ ,  $(x_0 > 0)$ ,  $\|a\|_{L^q(\mathbf{R}^+)} \leq (x_1 - x_0)^{1/q-1/p}$ , and  $\int_0^{\infty} a(x) dx = 0$ . Let us prove (12).

From the vanishing property of  $a$ , it is easy to see that  $\text{supp } Ha \subset (x_0, x_1)$ . Thus, noticing that  $x_0 > 0$ , we have

$$\begin{aligned}
\int_0^{\infty} |Ha(x)|^p dx &= \int_{x_0}^{x_1} |Ha(x)|^p dx \\
&= \int_{x_0}^{x_1} \left| \frac{1}{x} \int_{x_0}^x a(t) dt \right|^p dx \\
&\leq \int_{x_0}^{x_1} \left( \frac{1}{x} \left( \int_{x_0}^x |a(t)|^q dt \right)^{1/q} (x - x_0)^{1/q'} \right)^p dx \\
&\quad \text{(by Holder's inequality)} \\
&\leq \|a\|_{L^q(\mathbf{R}^+)}^p \int_{x_0}^{x_1} \left( \frac{1}{x} (x - x_0)^{1/q'} \right)^p dx \\
&\leq (x_1 - x_0)^{p/q-1} \int_{x_0}^{x_1} \frac{(x - x_0)^p}{x^p} (x - x_0)^{-p/q} dx \\
&< (x_1 - x_0)^{p/q-1} \int_{x_0}^{x_1} (x - x_0)^{-p/q} dx \\
&= 1/(1 - p/q)
\end{aligned}$$

when  $1 < q < \infty$ ;

$$\begin{aligned}
\int_0^{\infty} |Ha(x)|^p dx &= \int_{x_0}^{x_1} \left| \frac{1}{x} \int_{x_0}^x a(t) dt \right|^p dx \\
&\leq \|a\|_{L^q(\mathbf{R}^+)}^p \int_{x_0}^{x_1} \left( \frac{1}{x} \right)^p dx \\
&\leq (x_1 - x_0)^{p-1} \int_{x_0}^{x_1} \frac{(x - x_0)^p}{x^p} (x - x_0)^{-p} dx \\
&< (x_1 - x_0)^{p-1} \int_{x_0}^{x_1} (x - x_0)^{-p} dx \\
&= 1/(1 - p)
\end{aligned}$$

when  $q = 1$  and  $p \neq 1$ ; and

$$\begin{aligned}
\int_0^\infty |Ha(x)|^p dx &= \int_{x_0}^{x_1} \left| \frac{1}{x} \int_{x_0}^x a(t) dt \right|^p dx \\
&\leq \|a\|_{L^\infty(\mathbf{R}^+)}^p \int_{x_0}^{x_1} \left( \frac{1}{x} (x - x_0) \right)^p dx \\
&\leq (x_1 - x_0)^{-1} \int_{x_0}^{x_1} \frac{(x - x_0)^p}{x^p} dx \\
&< 1.
\end{aligned}$$

when  $q = \infty$ . Thus, (12) has been proved for  $1 \leq q \leq \infty$ , and  $p < q$ . Thus, we finish the proof of Proposition 1.

**Proof of Proposition 3** Let  $H^{p,q,0}(\mathbf{R}^+)$ , then  $f = \sum_i \lambda_i a_i$  where  $a_i$  are  $(p, q, 0)$ -atoms and  $\sum_i |\lambda_i|^p < +\infty$ . We know that  $Ha_i$  is well defined for every  $i$  since  $a_i \in L^q(\mathbf{R}^+)$  with  $1 \leq q < \infty$  and (12) holds by Proposition 1. Then

$$\left\| \sum_i \lambda_i Ha_i \right\|_{L^p(\mathbf{R}^+)}^p \leq \sum_i |\lambda_i|^p \|Ha_i\|_{L^p(\mathbf{R}^+)}^p \leq C \sum_i |\lambda_i|^p \leq C \|f\|_{H^{p,q,0}(\mathbf{R}^+)}^p < \infty,$$

it follows  $|\sum_i \lambda_i Ha_i(x)| < \infty$  a.e.. Let

$$f = \sum_i \lambda_i^{(1)} a_i^{(1)} = \sum_i \lambda_i^{(2)} a_i^{(2)} \quad (13)$$

with

$$\sum_i |\lambda_i^{(1)}|^{\bar{p}} < +\infty \quad \text{and} \quad \sum_i |\lambda_i^{(2)}|^{\bar{p}} < +\infty \quad (14)$$

and  $a_i^{(1)}$  and  $a_i^{(2)}$  are  $(p, s, \alpha)$ -atoms. Once it is proved that

$$\sum_i \lambda_i^{(1)} Ha_i^{(1)} = \sum_i \lambda_i^{(2)} Ha_i^{(2)} \quad \text{a.e.},$$

then,

$$Hf(x) = \sum_i \lambda_i Ha_i(x) \quad \text{a.e.}$$

is well defined for all  $f = \sum_{k=1}^\infty \lambda_k a_k \in H^{p,q,0}(\mathbf{R}^+)$ . Thus, Proposition 2 holds.

It is remained to prove (13). For any  $\delta > 0$ , by (14), there exists  $i_0$  such that

$$\sum_{i=i_0}^\infty |\lambda_i^{(1)}|^{\bar{p}} < \delta^{\bar{p}} \quad \text{and} \quad \sum_{i=i_0}^\infty |\lambda_i^{(2)}|^{\bar{p}} < \delta^{\bar{p}}. \quad (15)$$

From (13), we see that

$$\sum_{i=1}^{i_0-1} (\lambda_i^{(1)} a_i^{(1)} - \lambda_i^{(2)} a_i^{(2)}) = \sum_{i=i_0}^\infty \lambda_i^{(2)} a_i^{(2)} - \sum_{i=i_0}^\infty \lambda_i^{(1)} a_i^{(1)},$$

then,

$$\left\| \sum_{i=1}^{i_0-1} (\lambda_i^{(1)} a_i^{(1)} - \lambda_i^{(2)} a_i^{(2)}) \right\|_{H^{p,q,0}(\mathbf{R}^+)}^p \leq \sum_{i=i_0}^{\infty} |\lambda_i^{(1)}|^p + \sum_{i=i_0}^{\infty} |\lambda_i^{(2)}|^p < 2\delta^p. \quad (16)$$

By the linearity of  $H$ , we have

$$\sum_{i=1}^{\infty} \lambda_i^{(1)} H a_i^{(1)} - \sum_{i=1}^{\infty} \lambda_i^{(2)} H a_i^{(2)} = H \left( \sum_{i=1}^{i_0-1} (\lambda_i^{(1)} a_i^{(1)} - \lambda_i^{(2)} a_i^{(2)}) \right) + \sum_{i=i_0}^{\infty} \lambda_i^{(1)} H a_i^{(1)} - \sum_{i=i_0}^{\infty} \lambda_i^{(2)} H a_i^{(2)}. \quad (17)$$

By Proposition 2 and (16), we see that

$$\left\| H \left( \sum_{i=1}^{i_0-1} (\lambda_i^{(1)} a_i^{(1)} - \lambda_i^{(2)} a_i^{(2)}) \right) \right\|_{L^p(\mathbf{R}^+)}^p \leq \left\| \sum_{i=1}^{i_0-1} (\lambda_i^{(1)} a_i^{(1)} - \lambda_i^{(2)} a_i^{(2)}) \right\|_{H^{p,s}(\mathbf{R}^+)}^p < 2\delta^p. \quad (18)$$

From (17), (18), (12) and (15), we have

$$\left\| \sum_{i=1}^{\infty} \lambda_i^{(1)} T_{\varepsilon} a_i^{(1)} - \sum_{i=1}^{\infty} \lambda_i^{(2)} T_{\varepsilon} a_i^{(2)} \right\|_{L^p(\mathbf{R}^+)}^p < 4\delta^p.$$

Let  $\delta \rightarrow 0$ , we get that  $\left\| \sum_{i=1}^{\infty} \lambda_i^{(1)} H a_i^{(1)} - \sum_{i=1}^{\infty} \lambda_i^{(2)} H a_i^{(2)} \right\|_{L^p(\mathbf{R}^+)}^p = 0$ , it follows that  $\sum_{i=1}^{\infty} \lambda_i^{(1)} H a_i^{(1)} = \sum_{i=1}^{\infty} \lambda_i^{(2)} H a_i^{(2)}$  a.e.. Thus, we finish the proof of Proposition 3.

The proof of Theorem 1 is finished.

**Proof of Theorem 2** As the proof of Theorem 1, it suffices to prove the following propositions.

**Proposition 4** Let  $0 < p \leq 1$ . We have

$$\int_0^{\infty} |H^* a(x)|^p dx < \begin{cases} 1, & \text{if } q = \infty, \\ \frac{1}{(1-p)(p+1)^p}, & \text{if } q = 1, p \neq 1, \\ \frac{1}{(1-pq'/q)^{p/q'}} \frac{(1+p)^{p/q-1}}{(1/q' - p/q)p+1}, & \text{if } 1 < q < \infty, p < q-1, \end{cases} \quad (19)$$

for all  $(p, q, 0)_{x^p}$ -atom on  $\mathbf{R}^+$ .

Let  $FH_{x^p}^{p,q,0}(\mathbf{R}^+)$  be the set of all finite linear combination of  $(p, q, 0)_{x^p}$ -atoms.

From Proposition 4 it is easy to get that

**Proposition 5** Let  $p$  and  $q$  as in Theorem 2. Then (7) holds for all  $f \in FH_{x^p}^{p,q,0}(\mathbf{R}^+)$ .

**Proposition 6** Let  $0 < p \leq 1, 1 \leq q \leq \infty, p < q-1, p \neq q=1$  and  $p < q$ . Then

$$Hf(x) = \sum_{k=1}^{\infty} \lambda_k H a_k(x) \quad \text{a.e.}$$

for all  $f = \sum_{k=1}^{\infty} \lambda_k a_k \in H_{x^p}^{p,q,0}(\mathbf{R}^+)$ , where each  $a_k$  is a  $(p, q, 0)_{x^p}$ -atom on  $\mathbf{R}^+$  and  $\sum_{k=1}^{\infty} |\lambda_k|^p < \infty$ .

**Proof of Proposition 4** Let  $a$  be a  $(p, q, 0)_{x^p}$ -atom on  $\mathbf{R}^+$  with  $\text{supp } a \subset (x_0, x_1) \subset \mathbf{R}^+, (x_0 > 0)$ . Let us prove (19).

As those discussions in the proof of Theorem 1 we have  $\text{supp } H^*a \subset (x_0, x_1)$ . Thus, when  $q = \infty$ , noticing that  $\|a\|_{L^\infty(\mathbf{R}^+)} = \|a\|_{L_{t^p}^\infty(\mathbf{R}^+)}$ , by (10), we have

$$\begin{aligned}
\int_0^\infty |H^*a(x)|^p dx &= \int_{x_0}^{x_1} |H^*a(x)|^p dx \\
&= \int_{x_0}^{x_1} \left| \int_x^{x_1} a(t) dt \right|^p dx \\
&\leq \|a\|_{L_{t^p}^\infty(\mathbf{R}^+)}^p \int_{x_0}^{x_1} (x_1 - x)^p dx \\
&\leq \left( \int_{x_0}^{x_1} t^p dt \right)^{-1} \int_{x_0}^{x_1} (x_1 - x)^p dx \\
&\leq \frac{(x_1 - x_0)^{p+1}}{x_1^{p+1} - x_0^{p+1}} \\
&< 1;
\end{aligned}$$

when  $q = 1, 0 < p < 1$ , by (11), we have

$$\begin{aligned}
\int_0^\infty |H^*a(x)|^p dx &\leq \int_{x_0}^{x_1} \left| \int_x^{x_1} |a(t)| t^p \frac{1}{t^p} dt \right|^p dx \\
&\leq \|a\|_{L_{t^p}^1(\mathbf{R}^+)}^p \int_{x_0}^{x_1} \frac{1}{x^{p^2}} dx \\
&\leq \left( \int_{x_0}^{x_1} t^p dt \right)^{p-1} \int_{x_0}^{x_1} \frac{1}{x^{p^2}} dx \\
&= \frac{(1+p)^{1-p}}{(1-p^2)} \frac{x_1^{1-p^2} - x_0^{1-p^2}}{(x_1^{1+p} - x_0^{1+p})^{1-p}} \\
&< \frac{1}{(1-p)(1+p)^p} \frac{x_1^{1-p^2} - x_0^{1-p^2}}{x_1^{1-p^2} - x_0^{1-p^2}} \\
&= \frac{1}{(1-p)(1+p)^p};
\end{aligned}$$

and when  $1 < q < \infty, 0 < p < q - 1$ , we see that  $(p+1)/q < 1$ , and then  $pq'/q < 1$ , by (10) and (11), we have

$$\begin{aligned}
\int_0^\infty |H^*a(x)|^p dx &= \int_{x_0}^{x_1} \left| \int_x^{x_1} a(t) dt \right|^p dx \\
&\leq \int_{x_0}^{x_1} \left( \left( \int_x^{x_1} |a(t)|^q t^p dt \right)^{1/q} \left( \int_x^{x_1} \frac{1}{t^{pq'/q}} dt \right)^{1/q'} \right)^p dx \\
&\quad \text{(by Holder's inequality)} \\
&\leq \|a\|_{L_{t^p}^q(\mathbf{R}^+)}^p \left( \frac{1}{1 - pq'/q} \right)^{p/q'} \int_{x_0}^{x_1} (x_1^{1-pq'/q} - x^{1-pq'/q})^{p/q'} dx \\
&\leq \left( \frac{1}{1 - pq'/q} \right)^{p/q'} \left( \int_{x_0}^{x_1} t^p dt \right)^{1-p/q} \int_{x_0}^{x_1} ((x_1 - x)^{1-pq'/q})^{p/q'} dx \\
&= \left( \frac{1}{1 - pq'/q} \right)^{p/q'} \frac{(1+p)^{p/q-1}}{(1/q' - p/q)p + 1} \frac{(x_1 - x_0)^{(1/q' - p/q)p+1}}{(x_1^{p+1} - x_0^{p+1})^{1-p/q}}
\end{aligned}$$

$$\begin{aligned}
&< \left( \frac{1}{1 - pq'/q} \right)^{p/q'} \frac{(1+p)^{p/q-1}}{(1/q' - p/q)p + 1} \frac{(x_1 - x_0)^{(1/q' - p/q)p+1}}{(x_1 - x_0)^{(p+1)(1-p/q)}} \\
&= \frac{1}{(1 - pq'/q)^{p/q'}} \frac{(1+p)^{p/q-1}}{(1/q' - p/q)p + 1}.
\end{aligned}$$

Thus, (19) has been proved for  $1 \leq q \leq \infty$ , and  $p < q - 1$ .

Proposition 5 follows easily from Proposition 4. Proof of Proposition 6 is same as that of Proposition 3.

Thus, we finish the proof of Theorem 2.

**Proof of Theorem 3** Let  $a$  be a  $L - (p, q, s)$ -atom on  $\mathbf{R}^+$  with  $\text{supp } a \subset (x_0, x_1)$ ,  $(x_0 > 0)$ . Let us prove that  $\frac{1}{q'}Ha$  is a  $(p, q, s)$ -atom on  $\mathbf{R}^+$ . In fact,

- (i) we have already proved  $\text{supp } Ha \subset (x_0, x_1)$ ,
- (ii) when  $q = \infty$ , noticing that  $|Ha(x)| = \left| \frac{1}{x} \int_{x_0}^x a(t) dt \right| \leq \|a\|_{L^\infty(\mathbf{R}^+)} \times \frac{x - x_0}{x} < \|a\|_{L^\infty(\mathbf{R}^+)}$ , we have

$$\|Ha\|_{L^\infty(\mathbf{R}^+)} < \|a\|_{L^\infty(\mathbf{R}^+)} \leq (x_1 - x_0)^{-1/p};$$

and when  $1 < q < \infty$ , by the  $L^q(\mathbf{R}^+)$  boundedness of  $H$ , we have

$$\left\| \frac{1}{q'}Ha \right\|_{L^q(\mathbf{R}^+)} < \|a\|_{L^q(\mathbf{R}^+)} \leq (x_1 - x_0)^{1/q-1/p}, \quad (20)$$

- (iii) by the vanishing property of  $a$ , we have

$$\begin{aligned}
\int_0^{+\infty} x^\beta Ha(x) dx &= \int_{x_0}^{x_1} x^\beta Ha(x) dx \\
&= \int_{x_0}^{x_1} x^{\beta-1} \int_{x_0}^x a(t) dt dx \\
&= \int_{x_0}^{x_1} a(t) \int_t^{x_1} x^{\beta-1} dx dt \\
&= \begin{cases} \frac{1}{\beta} \int_{x_0}^{x_1} a(t) (x_1^\beta - t^\beta) dt, & \text{if } \beta = 1, 2, \dots, s, \\ \int_{x_0}^{x_1} a(t) (\ln x_1 - \ln t) dt, & \text{if } \beta = 0 \end{cases} \\
&= 0, \quad \text{if } \beta = 0, 1, 2, \dots, s.
\end{aligned}$$

Thus, combining (i), (ii) and (iii), we see that  $\frac{1}{q'}Ha$  is a  $(p, q, s)$ -atom on  $\mathbf{R}^+$ . Let  $f \in LH^{p,q,s}(\mathbf{R}^+)$ , then  $f = \sum_{j=1}^{\infty} \lambda_j a_j$ , where each  $a_j$  is a  $L - (p, q, s)$ -atom on  $\mathbf{R}^+$ . So we have  $\frac{1}{q'}Hf = \sum_{j=1}^{\infty} \lambda_j \frac{1}{q'}Ha_j$ , where each  $\frac{1}{q'}Ha_j$  is a  $(p, q, s)$ -atom on  $\mathbf{R}^+$ . And by the definition:  $\left\| \frac{1}{q'}Hf \right\|_{H^{p,q,s}(\mathbf{R}^+)} = \inf \left( \sum_{j=1}^{\infty} |\mu_j|^p \right)^{1/p}$ , where the infimum is taken over all the decompositions  $\frac{1}{q'}Hf = \sum_{j=1}^{\infty} \mu_j b_j$ , and each  $b_j$  is a  $(p, q, s)$ -atom, we have

$$\left\| \frac{1}{q'}Hf \right\|_{H^{p,q,s}(\mathbf{R}^+)} = \inf_{\substack{\frac{1}{q'}Hf = \sum_{j=1}^{\infty} \mu_j b_j, \\ \text{each } b_j \text{ is a } (p,q,s)\text{-atom}}} \left( \sum_{j=1}^{\infty} |\mu_j|^p \right)^{1/p}$$



$$\begin{aligned}
&\leq \inf_{\substack{\frac{1}{q'}Hf = \sum_{j=1}^{\infty} \lambda_j \frac{1}{q'}Ha_j, \\ \text{each } a_j \text{ is a } L-(p,q,s)\text{-atom}}} \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \\
&\quad \text{(since each } \frac{1}{q'}Ha_j \text{ is a } (p,q,s)\text{-atom)} \\
&= \inf_{\substack{f = \sum_{j=1}^{\infty} \lambda_j a_j, \\ \text{each } a_j \text{ is a } L-(p,q,s)\text{-atom}}} \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \\
&= \|f\|_{LHp,q,s(\mathbf{R}^+)}.
\end{aligned}$$

Then

$$\|Hf\|_{Hp,q,s(\mathbf{R}^+)} \leq q' \|f\|_{LHp,q,s(\mathbf{R}^+)}^p.$$

Thus, we finish the proof of Theorem 3.

**Proof of Theorem 4** Let  $a$  be a  $(p, q, s)_{x^p}$ -atom on  $\mathbf{R}^+$  with  $\text{supp } a \subset (x_0, x_1) \subset \mathbf{R}^+, (x_0 > 0)$ . Let us prove that

$$\begin{aligned}
&(1+p)^{-1/p} H^*a(x), & \text{if } q = \infty, \\
&(1-p)(1+p)^{1-1/p} H^*a(x), & \text{if } q = 1, p \neq 1, \\
&\text{and } \frac{1}{q} H^*a(x), & \text{if } 1 < q < \infty, 0 < p \leq 1,
\end{aligned} \tag{21}$$

are the  $(p, q, s-1)$ -atoms.

(i) Clearly,  $\text{supp } H^*a \subseteq (x_0, x_1)$ .

(ii) When  $q = \infty$ , and  $x \in (x_0, x_1)$ , noticing that  $\|a\|_{L_{tp}^\infty(\mathbf{R}^+)} = \|a\|_{L^\infty(\mathbf{R}^+)}$ , and by (10), we have

$$\begin{aligned}
|H^*a(x)| &\leq (x_1 - x) \|a\|_{L^\infty(\mathbf{R}^+)} \\
&= (x_1 - x) \|a\|_{L_{tp}^\infty(\mathbf{R}^+)} \\
&< (x_1 - x_0) \left( \int_{x_0}^{x_1} x^p dx \right)^{-1/p} \\
&= (1+p)^{1/p} \frac{(x_1 - x_0)}{(x_1^{p+1} - x_0^{p+1})^{1/p}} \\
&< (1+p)^{1/p} \frac{(x_1 - x_0)}{((x_1 - x_0)^{p+1})^{1/p}} \\
&= (1+p)^{1/p} (x_1 - x_0)^{-1/p}.
\end{aligned} \tag{22}$$

When  $q = 1, 0 < p < 1$ , by (10) and (11), we have,

$$\begin{aligned}
\int_0^\infty |H^*a(x)| dx &\leq \int_{x_0}^{x_1} \left| \int_x^{x_1} |a(t)| t^p \frac{1}{t^p} dt \right| dx \\
&\leq \|a\|_{L_{tp}^1(\mathbf{R}^+)} \int_{x_0}^{x_1} \frac{1}{x^p} dx
\end{aligned}$$

$$\begin{aligned}
&\leq \left( \int_{x_0}^{x_1} t^p dt \right)^{1-1/p} \int_{x_0}^{x_1} \frac{1}{x^p} dx \\
&= \frac{1}{(1-p)(1+p)^{1-1/p}} \frac{x_1^{1-p} - x_0^{1-p}}{(x_1^{1+p} - x_0^{1+p})^{1/p-1}} \\
&< \frac{1}{(1-p)(1+p)^{1-1/p}} \frac{(x_1 - x_0)^{1-p}}{(x_1 - x_0)^{(1+p)(1-p)/p}} \\
&= \frac{1}{(1-p)(1+p)^{1-1/p}} (x_1 - x_0)^{1-1/p}.
\end{aligned}$$

When  $1 < q < \infty$ , by (2), we have

$$\left\| \frac{1}{q} H^* a \right\|_{L^q(\mathbf{R}^+)} < \|a\|_{L^q(\mathbf{R}^+, t^q)} \leq (x_1 - x_0)^{1/q-1/p}. \quad (23)$$

(iii) By the vanishing property of  $a$ , we have

$$\begin{aligned}
\int_0^{+\infty} x^\beta H^* a(x) dx &= \int_{x_0}^{x_1} x^\beta H^* a(x) dx \\
&= \int_{x_0}^{x_1} x^\beta \int_x^{x_1} a(t) dt dx \\
&= \int_{x_0}^{x_1} a(t) \int_{x_0}^t x^\beta dx dt \\
&= \frac{1}{\beta+1} \int_{x_0}^{x_1} a(t) t^{\beta+1} dt \\
&= 0, \quad \text{if } \beta = 0, 1, 2, \dots, s-1,
\end{aligned}$$

when  $0 \leq \beta \leq s-1$ . Thus, (21) has been proved. Therefore, Theorem 4 follows from this by analogous arguments to those in the proof of Theorem 3.

### 3. Remarks

**Remark 1** If the functions  $f$  in Theorems 3 and 4 are finite linear combinations of corresponding atoms, i.e.  $f = \sum_{j=1}^{k_0} \lambda_j a_j$ , then

the " $\leq$ " in (8), (9) will be changed to " $<$ ".

In fact, for Theorem 3, suppose that there is  $k_0 < \infty$  such that

$$f = \sum_{j=1}^{k_0} \lambda_j a_j, \text{ where each } a_j \text{ is a } L - (p, q, s) - \text{atom on } \mathbf{R}^+,$$

then, by the proof of theorem 3, for each  $a_j, j = 1, 2, \dots, k_0$ , we have

$$\|H a_j\|_{L^q(\mathbf{R}^+)} < q' (x_1 - x_0)^{1/q-1/p},$$

and it follows that there is  $0 < \epsilon_j < 1$  such that

$$\|H a_j\|_{L^q(\mathbf{R}^+)} < (q' - \epsilon_j) (x_1 - x_0)^{1/q-1/p}.$$

Let  $\epsilon = \min\{\epsilon_1, \epsilon_2, \dots, \epsilon_{k_0}\}$ , then

$$\|Ha_j\|_{L^q(\mathbf{R}^+)} < (q' - \epsilon)(x_1 - x_0)^{1/q-1/p}.$$

As the arguments in the proof of Theorem 3, we have that each  $\frac{1}{q'-\epsilon}Ha_j$  is a  $(p, q, s)$ -atom on  $\mathbf{R}^+$ , and

$$\|Hf\|_{H^{p,q,s}(\mathbf{R}^+)} \leq (q' - \epsilon)\|f\|_{LH^{p,q,s}(\mathbf{R}^+)}^p < q'\|f\|_{LH^{p,q,s}(\mathbf{R}^+)}^p.$$

Similar arguments above are suitable for the case of Theorem 4.

**Remark 2** If we drop the restriction  $x_0 > 0$  in the definitions of Hardy spaces (in Definition 1), then

the " $<$ " in (6), (7) will be changed to " $\leq$ ".

This is because of that the " $<$ " in (12) and (19) will be changed to " $\leq$ " if  $x_0 = 0$  in the proofs of Theorems 1 and 2.

**Remark 3** If the functions  $f$  in Theorems 3 and 4 are finite linear combinations of corresponding atoms, i.e.  $f = \sum_{j=1}^{k_0} \lambda_j a_j$ , then, even if dropping the restriction  $x_0 > 0$  in the definitions of Hardy spaces (in Definition 1), we have that:

the " $\leq$ " in (8) will be changed to " $<$ "

when  $1 < q < \infty$ , and

the " $\leq$ " in (9) will be changed to " $<$ "

when  $1 < q \leq \infty$ . These follow from the analogous arguments to those of Remark 1, since the " $<$ " in (20), (23), and the first " $<$ " in (22) hold still when  $x_0 = 0$ .

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